Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let $T_n(x)$ be polynomial defined by recurrence $T_{n+1} - 2xT_n - T_{n-1}, n \in \mathbb{N}$ with initial conditions $T_0 = 1, T_1 = x$. (First Kind Chebishev's Polynomials). Prove that

$$\sqrt[n]{T_n(x)} \leq 1 + n(x-1) \ , \ x \geq 1, n \in \mathbb{N};$$

Solution

Since $(1 + n(x-1))^n = \sum_{k=0}^{n} n^k \binom{n}{k} (x-1)^k$ then for the proof of inequality

(1) $\quad T_n(x) \leq (1 + n(x-1))^n$ convenient to use Taylor's representation of $T_n(x)$:

$T_n(x) = \sum_{k=0} \dfrac{T_n^{(k)}(1)}{k!}(x-1)^k$, because suffice to prove that $\dfrac{T_n^{(k)}(1)}{k!} \leq \binom{n}{k} n^k \Leftrightarrow$

(2) $\quad T_n^{(k)}(1) \leq n^k \cdot \dfrac{n!}{(n-k)!}$ where $k = 0, 1, \ldots, n$.

For calculation $T_n^{(k)}(1)$ we will partake derivative equation which define $n - th$ Chebishev's Polynomial $T_n(x)$:

(3) $\quad (1 - x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0.$

In the supposition that for arbitrary $k = 1, 2, \ldots, n-1$ consecutive derivatives $T_n^{(k+1)}(x), T_n^{(k)}(x), T_n^{(k-1)}$ satisfy to correlation

$(1 - x^2)T_n^{(k+1)}(x) - a_k x T_n^{(k)}(x) + b_k T_n^{(k-1)}(x) = 0$ we obtain that

$-2xT_n^{(k+1)}(x) + (1 - x^2)T_n^{(k+2)}(x) - a_k T_n^{(k)}(x) - a_k x T_n^{(k+1)}(x) + b_k T_n^{(k)}(x) = 0 \Leftrightarrow$

$(1 - x^2)T_n^{(k+2)}(x) - (a_k + 2)xT_n^{(k+1)}(x) + (b_k - a_k)T_n^{(k)}(x) = 0.$

Thus we have $a_{k+1} = a_k + 2$ and $b_{k+1} = b_k - a_k$ where $a_1 = 1$ and $b_1 = n^2$.

Hence, $a_k = 2k - 1$ and $b_{k+1} - b_1 = \sum_{i=1}^{k}(b_{i+1} - b_i) = -\sum_{i=1}^{k}(2i-1) = -k^2$ and therefore $a_{k+1} = 2k + 1, b_{k+1} = n^2 - k^2$.

So, for $T_n^{(k+2)}(x), T_n^{(k+1)}(x), T_n^{(k)}$ we obtain following correlation

(4) $(1 - x^2)T_n^{(k+2)}(x) - (2k+1)xT_n^{(k+1)}(x) + (n^2 - k^2)T_n^{(k)}(x) = 0$, where $k = 0, 1, \ldots, n-1$ and in particularly for $x = 1$ we have:

(5) $(2k+1)T_n^{(k+1)}(1) = (n^2 - k^2)T_n^{(k)}(1) \Leftrightarrow \dfrac{T_n^{(k+1)}(1)}{T_n^{(k)}(1)} = \dfrac{n^2 - k^2}{2k+1}.$

For $k = 0$ inequality (2) obviously holds because $T_n^{(0)}(1) = T_n(1) = 1$.

Since $\dfrac{T_n^{(k+1)}(1)}{T_n^{(k)}(1)} \leq \dfrac{n^{k+1} \cdot \dfrac{n!}{(n-k-1)!}}{n^k \cdot \dfrac{n!}{(n-k)!}} \Leftrightarrow \dfrac{n^2 - k^2}{2k+1} \leq n(n-k) \Leftrightarrow$

$n + k \leq (2k+1)n$ then, from of Math. Induction's supposition that

$$T_n^{(k)}(1) \leq n^k \cdot \frac{n!}{(n-k)!} \ ,$$

we immediately obtain that $T_n^{(k+1)}(1) = T_n^{(k)}(1) \cdot \dfrac{T_n^{(k+1)}(1)}{T_n^{(k)}(1)} \leq$

$$n^k \cdot \frac{n!}{(n-k)!} \cdot n(n-k) = n^{k+1} \cdot \frac{n!}{(n-(k+1))!}.$$

Equality in (1) occurs iff $n = 1$ and don't holds if $n > 1$. (because

$T_n(x) = (1 + n(x-1))^n, x \geq 1 \Leftrightarrow T_n^{(k)}(1) = n^k \bullet \dfrac{n!}{(n-k)!}, k = 0, 1, \ldots, n \Leftrightarrow T_n(1) = 1$ and $\dfrac{T_n^{(k+1)}(1)}{T_n^{(k)}(1)} = n(n-k), k = 1, \ldots, n-1 \Leftrightarrow \dfrac{n^2 - k^2}{2k+1} = n(n-k), k = 1, \ldots, n-1 \Leftrightarrow$ $n + k = (2k+1)n, k = 1, \ldots, n-1 \Leftrightarrow (2n-1)k = 0$).